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Spectral representation of the pentagon diagram amplitude*

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A method developed in two previous papers is used to derive a double spectral representation with Mandelstam boundary for the pentagon diagram amplitude for the production process $A\ B \rightarrow C\ D\ N$. Restrictions on the masses and kinematic invariants for which this representation is valid are found and it is discussed how a representation can be obtained for wider ranges of these variables. Finally, a comparison is made with the results of other authors.

1. INTRODUCTION

Different aspects of the properties of the pentagon diagram amplitude or five-point function have been discussed by a number of authors. Cutkosky used the Landau–Cutkosky rules[1,2] to show that, unlike the leading singularities of the triangle and box diagram amplitudes, the leading singularity of the pentagon diagram amplitude is not a branch point. The discontinuity associated with this singularity, as calculated by the Cutkosky rules, is a delta function.[3] Cook and Tarski[4] made a detailed study of the leading Landau curve of the pentagon diagram amplitude and determined the singular points of this amplitude for several specific processes. A reduction formula expressing the pentagon diagram amplitude in terms of five box daigram amplitudes was obtained by Halpern.[5]

The pentagon diagram amplitude has also been studied with a view to writing it as a double spectral representation, for a restricted range of masses and kinematic invariants, by Zav'yalov and Pavlov.[6] Their analysis however contains a number of errors. In particular, the double spectral representation obtained by them [Eq. (23) of Ref. 6] is divergent, that is, infinity is obtained when the integration is carried out. Further, the properties of the roots of the quadratic equation yielding the leading Landau curve of the pentagon diagram amplitude are more complicated than indicated in Ref. 6. The roots can under certain circumstances becomes complex and this is another reason why their spectral representation is incorrect.

In this paper we extend a method used in two previous papers, Ref. 7 (referred to as VF) and Ref. 8 (referred to as I), to obtain a double spectral representation for the pentagon diagram amplitude, for a restricted range of masses and kinematic invariants. (Equations from I will be denoted by placing an I- in front of the equation number).

In Sec. 2, the pentagon diagram amplitude associated with the pentagon diagram in Fig. 1 is transformed from its Feynman parametrized form into a more convenient form and the restrictions made on the values of the masses and kinematic invariants are discussed. The boundary of the region of integration in the quadruple integral obtained in Sec. 2 is studied in Sec. 3 and in Sec. 4 we obtain some results necessary for reversing the order of integration.

The order of integration is reversed in Sec. 5 and a triple integral representation is obtained. In Sec. 6 the boundary of the region of integration in the triple integral is studied and in Sec. 7 results necessary for reversing the order of integration are obtained. Finally, in Sec. 8 the order of integration is reversed and an integration is carried out to obtain a double spectral representation in s and t for the pentagon diagram amplitude. We also note in Sec. 8 that one of the integrations can be carried out to obtain a single dispersion relation in s and in principle the method of Ref. 9 (referred to as II) can be used to obtain a representation for the pentagon diagram amplitude for general physical invariants. Using this method it should be possible to determine directly how and when complex triangle, box, and pentagon singularities occur, resulting in a breakdown of even a single dispersion integral over a real domain.

2. TRANSFORMATION OF THE PENTAGON DIAGRAM AMPLITUDE

With plane wave states normalized so that $\langle p' | p\rangle = \delta^{(3)}(p' - p)$ we define the scalar invariant production amplitude $P(s_1, s_2, s_3, s_4, s_5)$ for the process $AB \rightarrow CDN$ in terms of the S-operator by

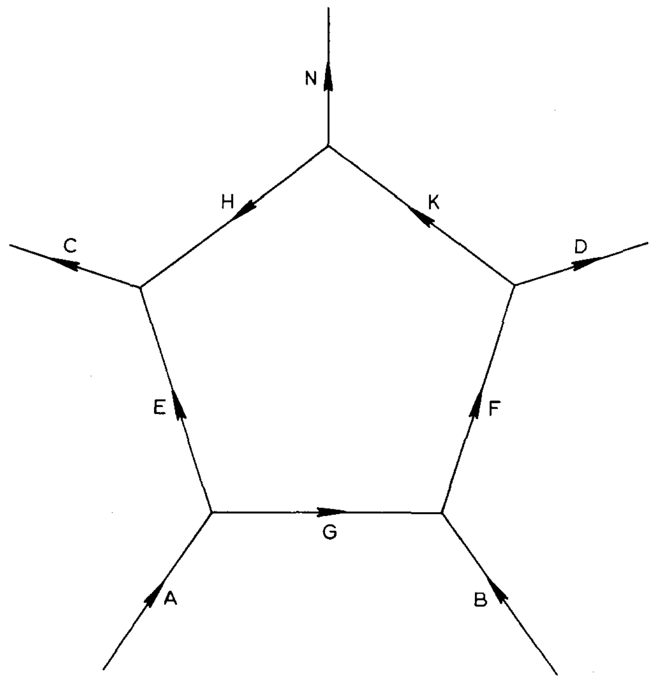

FIG. 1. Pentagon amplitude for the production process $AB \rightarrow CDN$.

$$\langle \mathbf{p}_C \mathbf{p}_D \mathbf{p}_N | S | \mathbf{p}_A \mathbf{p}_B \rangle = -i(2\pi)^4 \delta^{(4)}(p_C + p_D + p_N - p_A - p_B)(2\pi)^{-15/2} \times (2E_A)^{-1/2}(2E_B)^{-1/2}(2E_C)^{-1/2}(2E_D)^{-1/2}(2E_N)^{-1/2} \times P(s_1, s_2, s_3, s_4, s_5), \quad (1)$$

where $s_1 = (p_A + p_B)^2$, $s_2 = (p_A - p_C)^2$, $s_3 = (p_B - p_D)^2$, $s_4 = (p_D + p_N)^2$, $s_5 = (p_C + p_N)^2$ are five independent kinematic invariants. (The notation has been chosen so that the results of I can be applied without the need to relabel the variables.) Then, using standard Feynman rules[10] and the Feynman identity, we find that the amplitude arising from the pentagon diagram of Fig. 1 takes the form

$$P_{\text{pent.}}(s_1, s_2, s_3, s_4, s_5) = -(g/64\pi^2 EFGH) I(x_1, x_2, x_3, x_4, x_5), \quad (2)$$

where, writing $I(x_i)$ for $I(x_1, x_2, x_3, x_4, x_5)$,

$$I(x_i) = -4EFGH \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \int_0^{1-\alpha-\beta} d\gamma \int_0^{1-\alpha-\beta-\gamma} d\delta \times [E^2\alpha + F^2\beta + G^2(1-\alpha-\beta-\gamma-\delta) + H^2\gamma + K^2\delta - A^2(1-\alpha-\beta-\gamma-\delta)\alpha - B^2(1-\alpha-\beta-\gamma-\delta)\beta - C^2\alpha\gamma - D^2\beta\delta - N^2\gamma\delta - s_1\alpha\beta - s_2(1-\alpha-\beta-\gamma-\delta)\gamma - s_3(1-\alpha-\beta-\gamma-\delta)\delta \cdot s_4\beta\gamma - s_5\alpha\delta]^{-3}. \quad (3)$$

In Eq. (3), the s_i are to be expressed in terms of the new variables x_i or X_i defined by

$$\begin{aligned} x_1 &= -X_1 = (2EF)^{-1}(s_1 - E^2 - F^2),\\ x_2 &= -X_2 = (2GH)^{-1}(s_2 - G^2 - H^2),\\ x_3 &= -X_3 = (2GK)^{-1}(s_3 - G^2 - K^2),\\ x_4 &= -X_4 = (2FH)^{-1}(s_4 - F^2 - H^2),\\ x_5 &= -X_5 = (2EK)^{-1}(s_5 - E^2 - K^2). \end{aligned} \quad (4)$$

We shall find it convenient to use both the quantities x_i and X_i $(i = 1, \ldots, 5)$ in the following. The factor g in Eq. (2) is given by $g = g_{AEG}\, g_{BFG}\, g_{CEH}\, g_{DFK}\, g_{NHK}$, where $g_{AEG}, \ldots, g_{NHK}$ are the usual rationalized coupling constants.

We begin by generalizing the transformation used in I. The change of variables is

$$\lambda = (\alpha+\beta)^{-1}(1-\alpha-\beta-\gamma-\delta), \quad \mu = (\alpha+\beta)^{-1}\gamma,$$
$$\zeta = (\alpha+\beta)^{-1}\delta, \quad \nu = \beta^{-1}(\alpha+\beta),$$

with the inverse

$$\alpha = \nu^{-1}(\nu-1)(1+\lambda+\mu+\zeta)^{-1}, \quad \beta = \nu^{-1}(1+\lambda+\mu+\zeta)^{-1},$$
$$\gamma = \mu(1+\lambda+\mu+\zeta)^{-1}, \quad \delta = \zeta(1+\lambda+\mu+\zeta)^{-1}.$$

The Jacobian of the transformation is given by

$$|\partial(\alpha,\beta,\gamma,\delta)/\partial(\lambda,\mu,\nu,\zeta)| = (1+\lambda+\mu+\zeta)^{-5}\nu^{-2}$$

and we find that

$$\begin{aligned} I(x_i) = &-4EFGH \int_0^\infty d\zeta \int_0^\infty d\mu \int_0^\infty d\lambda \int_0^\infty d\nu\, (1+\lambda+\mu+\zeta)\nu^{-2}\\ &\times [E^2\nu^{-1}(\nu-1)(1+\lambda+\mu+\zeta) + F^2\nu^{-1}(1+\lambda+\mu+\zeta)\\ &+ G^2\lambda(1+\lambda+\mu+\zeta) + H^2\mu(1+\lambda+\mu+\zeta)\\ &+ K^2\zeta(1+\lambda+\mu+\zeta)\\ &- A^2\lambda\nu^{-1}(\nu-1) - B^2\lambda\nu^{-1} - C^2\mu\nu^{-1}(\nu-1) - D^2\zeta\nu^{-1} - N^2\mu\zeta\\ &- s_1\nu^{-2}(\nu-1) - s_2\lambda\mu - s_3\lambda\zeta - s_4\nu^{-1}\mu - s_5\nu^{-1}(\nu-1)\zeta]^{-3}\\ = &\,2EFGH \int_0^\infty \frac{d\zeta}{\zeta} \int_0^\infty d\mu \int_0^\infty d\lambda \int_1^\infty d\nu.\\ &\times \frac{\partial}{\partial K^2}[(\nu-1)\phi(\lambda,\mu,\zeta) + \psi(\lambda,\mu,\zeta) - \nu^{-1}(\nu-1)v(x_1)]^{-2}, \end{aligned} \quad (5)$$

where

$$\phi(\lambda,\mu,\zeta) = G^2\lambda^2 + H^2\mu^2 + K^2\zeta^2 + 2GHX_2\lambda\mu + 2GKX_3\lambda\zeta + 2HKe\mu\zeta + 2EGa\lambda + 2EHc\mu + 2EKX_5\zeta + E^2, \quad (6)$$

$$\psi(\lambda,\mu,\zeta) = G^2\lambda^2 + H^2\mu^2 + K^2\zeta^2 + 2GHX_2\lambda\mu + 2GKX_3\lambda\zeta + 2HKe\mu\zeta + 2FGb\lambda + 2FHX_4\mu + 2FKd\zeta + F^2, \quad (7)$$

$$v(x_1) = 2EFx_1 + E^2 + F^2. \quad (8)$$

The constants A^2, B^2, C^2, D^2, N^2 have been expressed in terms of a, b, c, d, e defined by

$$\begin{aligned} &2EGa = E^2 + G^2 - A^2, \quad 2FGb = F^2 + G^2 - B^2,\\ &2EHc = E^2 + H^2 - C^2, \quad 2FKd = F^2 + K^2 - D^2,\\ &2HKe = H^2 + K^2 - N^2, \end{aligned} \quad (9)$$

and we have also used Eq. (4).

To simplify the proof of a spectral representation we restrict the quantities defined in Eqs. (4) and (9) as follows:

$$a, b, c, d, e > 0, \quad X_i > 0 \quad (i = 1, \ldots, 5). \quad (10)$$

Equation (10) ensures that $\phi(\lambda,\mu,\zeta) > 0$, $\psi(\lambda,\mu,\zeta) > 0$ for $\lambda \geq 0$, $\mu \geq 0$, $\zeta \geq 0$; in fact the term in square brackets in Eq. (5) is always positive and $I(x_i)$ is well defined. While Eq. (10) can be satisfied with physical invariants by choosing the internal masses sufficiently large, the restrictions on X_i mean that the amplitude we are considering does not in general correspond to a physical process since for the physical amplitude associated with the pentagon diagram in Fig. 1 X_1, X_4, and X_5 would in general be negative. However, from the form of $I(x_i)$ we see that a spectral representation cannot in general be proved for negative X_1, X_4, and X_5, using real analysis only. One way of obtaining the physical amplitude would be to start with the spectral representation for the unphysical amplitude [Eq. (65)] and do an analytic continuation in X_1, X_4, and X_5 using, for example, a generalization of the method used in II. We discuss this problem further in Sec. 8.

The argument leading to Eqs. (I-19) and (I-20) can now be used to show that

$$I(x_i) = \lim_{\epsilon \downarrow 0} \frac{\partial}{\partial K^2} \int_\epsilon^\infty \frac{d\zeta}{\zeta} \int_0^\infty \frac{d\mu}{\mu} \times \lim_{\delta \downarrow 0} \frac{\partial}{\partial x_2} \int_\delta^\infty \frac{d\lambda}{\lambda} \int_{h(\lambda,\mu,\zeta)}^\infty \frac{d\xi}{(\xi - x_1)[U(\xi,\lambda,\mu,\zeta)]^{1/2}}, \quad (11)$$

where

$$U(\xi,\lambda,\mu,\zeta) = (\xi - h(\lambda,\mu,\zeta))(\xi - k(\lambda,\mu,\zeta)), \quad (12)$$

$$\left.\begin{matrix} h(\lambda,\mu,\zeta) \\ k(\lambda,\mu,\zeta) \end{matrix}\right\} = (2EF)^{-1}\{[\sqrt{\phi(\lambda,\mu,\zeta)} \pm \sqrt{\psi(\lambda,\mu,\zeta)}]^2 - E^2 - F^2\}, \quad (13)$$

and $\phi(\lambda, \mu, \zeta)$, $\psi(\lambda, \mu, \zeta)$ are given by Eqs. (6), (7).

3. STUDY OF $h(\lambda,\mu,\zeta)$

To reverse the order of integration in Eq. (11) we need to examine the function $h(\lambda, \mu, \zeta)$ for $\lambda \geq 0$, $\mu \geq 0$, $\zeta \geq 0$. As in Sec. 4 of I (or of VF) we write

$$\phi(\lambda, \mu, \zeta) = p_1\lambda^2 + 2q_1(\mu, \zeta)\lambda + r_1(\mu, \zeta),$$
$$\psi(\lambda, \mu, \zeta) = p_1\lambda^2 + 2q_1'(\mu, \zeta)\lambda + r_1'(\mu, \zeta), \tag{14}$$

where

$$p_1 = G^2,$$
$$q_1(\mu, \zeta) = G(HX_2\lambda + KX_3\zeta + Ea),$$
$$r_1(\mu, \zeta) = H^2\mu^2 + K^2\zeta^2 + 2HKe\mu\zeta + 2EHc\mu + 2EKX_5\zeta + E^2,$$
$$q_1'(\mu, \zeta) = G(HX_2\lambda + KX_3\zeta + Fb),$$
$$r_1'(\mu, \zeta) = H^2\mu^2 + K^2\zeta^2 + 2HKe\mu\zeta + 2FHX_4\mu + 2FKd\zeta + F^2. \tag{15}$$

Then the argument of Sec. 4 of I (or of VF) shows that for fixed $\mu \geq 0$, $\zeta \geq 0$, $h(\lambda, \mu, \zeta)$ increases strictly from $h(0, \mu, \zeta)$ to $+\infty$ as λ increases from 0 to $+\infty$, whenever $h_\lambda(0, \mu, \zeta) \geq 0$. Now

$$h_\lambda(0, \mu, \zeta) = (EF)^{-1}(\sqrt{r_1(\mu, \zeta)} + \sqrt{r_1'(\mu, \zeta)})\, l_1(\mu, \zeta), \tag{16}$$

where

$$l_1(\mu, \zeta) = [q_1(\mu, \zeta)/\sqrt{r_1(\mu, \zeta)}] + [q_1'(\mu, \zeta)/\sqrt{r_1'(\mu, \zeta)}]. \tag{17}$$

Thus, when Eq. (10) holds, it follows from Eqs. (15), (17), and (16) that for fixed $\mu \geq 0$, $\zeta \geq 0$, $h(\lambda, \mu, \zeta)$ increases strictly from $h(0, \mu, \zeta)$ to $+\infty$ as λ increases from 0 to $+\infty$. Similarly, for fixed $\lambda \geq 0$, $\zeta \geq 0$, $h(\lambda, \mu, \zeta)$ increases strictly from $h(\lambda, 0, \zeta)$ to $+\infty$ as μ increases from 0 to $+\infty$ and for fixed $\mu \geq 0$, $\zeta \geq 0$, $h(\lambda, \mu, \zeta)$ increases strictly from $h(\lambda, \mu, 0)$ to $+\infty$ as ζ increases from 0 to $+\infty$.

4. SOLUTIONS OF $U(\xi,\lambda,\mu,\zeta) = 0$

In this section we study the behavior of the zeros of $U(\xi, \lambda, \mu, \zeta)$ first when ξ, μ and ζ are held fixed, then when ξ, λ, and ζ are held fixed and finally when ξ, λ, and μ are held fixed. From Eqs. (12), (13), (6), and (7) we have

$$\begin{aligned}4E^2F^2\, U(\xi, \lambda, \mu, \zeta) &= a_1(\xi)\lambda^2 + 2b_1(\xi, \mu, \zeta)\lambda + c_1(\xi, \mu, \zeta)\\ &= a_2(\xi)\mu^2 + 2b_2(\xi, \lambda, \zeta)\mu + c_2(\xi, \lambda, \zeta)\\ &= a_3(\xi)\zeta^2 + 2b_3(\xi, \lambda, \mu)\zeta + c_3(\xi, \lambda, \mu),\end{aligned} \tag{18}$$

where

$$a_1(\xi) = 4G^2[(Ea - Fb)^2 - v(\xi)],$$
$$a_2(\xi) = 4H^2[(Ec - FX_4)^2 - v(\xi)],$$
$$a_3(\xi) = 4K^2[(EX_5 - Fd)^2 - v(\xi)],$$
$$b_1(\xi, \mu, \zeta) = \beta(\xi, -X_2)\mu + \gamma(\xi, -X_3)\zeta + b_1,$$
$$b_2(\xi, \lambda, \zeta) = \beta(\xi, -X_2)\lambda + \delta(\xi, -e)\zeta + b_2,$$
$$b_3(\xi, \lambda, \mu) = \gamma(\xi, -X_3)\lambda + \delta(\xi, -e)\zeta + b_3,$$
$$\beta(\xi, -X_2) = 4GH[(Ea - Fb)(Ec - FX_4) - X_2\, v(\xi)],$$
$$\gamma(\xi, -X_3) = 4GK[(Ea - Fb)(EX_5 - Fd) - X_3\, v(\xi)],$$
$$\delta(\xi, -e) = 4HK[(Ec - FX_4)(EX_5 - Fd) - e\, v(\xi)],$$
$$b_1 \equiv b_1(\xi, 0, 0) = 2G[(Ea - Fb)(E^2 - F^2) - (Ea + Fb)v(\xi)],$$
$$b_2 \equiv b_2(\xi, 0, 0) = 2H[(Ec - FX_4)(E^2 - F^2) - (Ec + FX_4)v(\xi)],$$
$$b_3 \equiv b_3(\xi, 0, 0) = 2K[(EX_5 - Fd)(E^2 - F^2) - (EX_5 + Fd)v(\xi)], \tag{19}$$

and $v(\xi)$ is given in Eq. (8).

The quantities $c_1(\xi, \mu, \zeta)$, $c_2(\xi, \lambda, \zeta)$, $c_3(\xi, \lambda, \mu)$ are determined from Eq. (18) by putting λ, μ, ζ, respectively, equal to zero and using in addition Eq. (19) and the fact that

$$c_1(\xi, 0, 0) = c_2(\xi, 0, 0) = c_3(\xi, 0, 0) = 4E^2F^2(\xi^2 - 1). \tag{20}$$

The argument of Sec. 5 of VF (see also Sec. 5 of I) shows that for each $\xi \geq h(0, \mu, \zeta)$, where μ and ζ are fixed and ≥ 0, the quadratic equation in λ

$$U(\xi, \lambda, \mu, \zeta) = 0$$

has two real roots given by

$$\lambda_\pm(\xi, \mu, \zeta) = [a_1(\xi)]^{-1}(-b_1(\xi, \mu, \zeta) \mp \{[b_1(\xi, \mu, \zeta)]^2 - a_1(\xi)c_1(\xi, \mu, \zeta)\}^{1/2}). \tag{21}$$

From Eqs. (19), (13), (14), and (15) we see that

$$b_1(h(0, \mu, \zeta), \mu, \zeta) = -4[\sqrt{r_1(\mu, \zeta)} + \sqrt{r_1'(\mu, \zeta)}]\, \sqrt{r_1(\mu, \zeta)}\sqrt{r_1'(\mu, \zeta)}\, l_1(\mu, \zeta), \tag{22}$$

where $l_1(\mu, \zeta)$ is given in Eq. (17). Since $l_1(\mu, \zeta) > 0$ when Eq. (10) holds it follows that $\lambda_+(h(0, \mu, \zeta), \mu, \zeta) = 0 \neq \lambda_-(h(0, \mu, \zeta), \mu, \zeta)$ and in fact $\lambda_+(\xi, \mu, \zeta)$ is the inverse of the strictly increasing function $h(\lambda, \mu, \zeta)$ on $0 \leq \lambda < \infty$. Thus $\lambda_+(\xi, \mu, \zeta)$ increases strictly from 0 to $+\infty$ as ξ increases from $h(0, \mu, \zeta)$ to $+\infty$. Similarly for each $\xi \geq h(\lambda, 0, \zeta)$, where λ and ζ are fixed and ≥ 0, the quadratic equation in μ

$$U(\xi, \lambda, \mu, \zeta) = 0$$

has two real roots, $\mu_\pm(\xi, \lambda, \zeta)$ given by the right-hand side of Eq. (21) with $\mu \to \lambda$, $1 \to 2$. The root $\mu_+(\xi, \lambda, \zeta)$ is the inverse of the strictly increasing function $h(\lambda, \mu, \zeta)$ on $0 \leq \mu < \infty$. Further, for each $\xi \geq h(\lambda, \mu, 0)$, where λ and μ are fixed and ≥ 0, the quadratic equation in ζ

$$U(\xi, \lambda, \mu, \zeta) = 0$$

has two real roots, $\zeta_\pm(\xi, \lambda, \mu)$ given by the right-hand side of Eq. (21) with $\mu \to \lambda$, $\zeta \to \mu$, $1 \to 3$. Again $\zeta_+(\xi, \lambda, \mu)$ is the inverse of the strictly increasing function $h(\lambda, \mu, \zeta)$ on $0 \leq \zeta < \infty$.

5. REVERSAL OF ORDER OF INTEGRATION

Since Eq. (10) holds, we showed in Sec. 3 that $h_\lambda(0, \mu, \zeta) > 0$ for all $\mu \geq 0$, $\zeta \geq 0$ and so from Sec. 4, $\lambda_+(\xi, \mu, \zeta)$ is the inverse of the strictly increasing function $h(\lambda, \mu, \zeta)$ on $0 \leq \lambda < \infty$ for each $\mu \geq 0$, $\zeta \geq 0$. Thus Eq. (11) can be written

$$I(x_i) = 2EF \lim_{\epsilon \downarrow 0} \frac{\partial}{\partial K^2} \int_\epsilon^\infty \frac{d\zeta}{\zeta} \int_0^\infty \frac{d\mu}{\mu} \times \lim_{\delta \downarrow 0} \frac{\partial}{\partial x_2} \int_{h(\delta, \mu, \zeta)}^\infty \frac{d\xi}{\xi - x_1}\, \Lambda(\xi, \delta, \mu, \zeta), \tag{23}$$

where

$$\Lambda(\xi,\delta,\mu,\zeta)=\int_{\delta}^{\lambda_+(\xi,\mu,\zeta)}\frac{d\lambda}{\lambda[a_1(\xi)\lambda^2+2b_1(\xi,\mu,\zeta)\lambda+c_1(\xi,\mu,\zeta)]^{1/2}}. \tag{24}$$

Note that $h(\delta,\mu,\zeta)$, $b_1(\xi,\mu,\zeta)$, $\lambda_+(\xi,\mu,\zeta)$, and $\Lambda(\xi,\delta,\mu,\zeta)$ depend on x_2.

Now since for fixed $\zeta\geqslant 0$, $\mu_+(\xi,0,\zeta)$ is the inverse of the strictly increasing function $h(0,\mu,\zeta)$ on $0\leqslant\mu<\infty$, the argument of Sec. 6 of I can be used to show that

$$I(x_i)=\lim_{\epsilon\downarrow 0}\frac{\partial}{\partial K^2}\int_{\epsilon}^{\infty}\frac{d\zeta}{\zeta}\int_{h(0,0,\zeta)}^{\infty}\frac{d\xi}{\xi-x_1}X(\xi,\zeta), \tag{25}$$

where

$$X(\xi,\zeta)=\frac{16EFGH\,v(\xi)}{[c_2(\xi,0,\zeta)]^{1/2}}\int_0^1\frac{du(lu^2-m)}{(lu^2-m)^2+4nu^2}. \tag{26}$$

Here

$$l=[\mu_-(\xi,0,\zeta)]^{-1}b_1(\xi,\mu_-(\xi,0,\zeta),\zeta)$$
$$m=[\mu_+(\xi,0,\zeta)]^{-1}b_1(\xi,\mu_+(\xi,0,\zeta),\zeta)$$
$$n=-a_1(\xi)[c_2(\xi,0,\zeta)]^{-1}\{[b_2(\xi,0,\zeta)]^2-a_2(\xi)c_2(\xi,0,\zeta)\}. \tag{27}$$

Since from Eq. (18) $c_1(\xi,0,\zeta)=c_2(\xi,0,\zeta)$, we have

$$\begin{aligned}lm-n&=[c_2(\xi,0,\zeta)]^{-2}([\overline{G}(\xi,x_2,\zeta)]^2\\&\quad-\{[b_1(\xi,0,\zeta)]^2-a_1(\xi)c_1(\xi,0,\zeta)\}\\&\quad\times\{[b_2(\xi,0,\zeta)]^2-a_2(\xi)c_2(\xi,0,\zeta)\})\\&=[c_2(\xi,0,\zeta)]^{-1}\overline{F}(\xi,x_2,\zeta),\end{aligned} \tag{28}$$

where

$$\overline{G}(\xi,x_2,\zeta)=-[\beta(\xi,x_2)c_2(\xi,0,\zeta)-b_1(\xi,0,\zeta)b_2(\xi,0,\zeta)], \tag{29}$$

and

$$\begin{aligned}\overline{F}(\xi,x_2,\zeta)&=[\beta(\xi,x_2)]^2c_2(\xi,0,\zeta)-2\beta(\xi,x_2)b_1(\xi,0,\zeta)b_2(\xi,0,\zeta)\\&\quad+[b_1(\xi,0,\zeta)]^2a_2(\xi)+[b_2(\xi,0,\zeta)]^2a_1(\xi)-a_1(\xi)a_2(\xi)c_2(\xi,0,\zeta)\\&=16G^2H^2[v(\xi)]^2c_2(\xi,0,\zeta)[x_2-f_+(\xi,\zeta)][x_2-f_-(\xi,\zeta)].\end{aligned} \tag{30}$$

Here

$$\begin{aligned}f_\pm(\xi,\zeta)&=[4GH\,v(\xi)c_2(\xi,0,\zeta)]^{-1}(-4GH\,c_2(\xi,0,\zeta)(Ea-Fb)\\&\quad\times(Ec-FX_4)+b_1(\xi,0,\zeta)b_2(\xi,0,\zeta)\pm\{[b_1(\xi,0,\zeta)]^2\\&\quad-a_1(\xi)c_1(\xi,0,\zeta)\}^{1/2}\\&\quad\times\{[b_2(\xi,0,\zeta)]^2-a_2(\xi)c_2(\xi,0,\zeta)\}^{1/2})\end{aligned} \tag{31}$$

and the argument of Sec. 5 of I (or of VF) shows that

$$\begin{aligned}&[b_1(\xi,0,\zeta)]^2-a_1(\xi)c_1(\xi,0,\zeta)>0,\\&[b_2(\xi,0,\zeta)]^2-a_2(\xi)c_2(\xi,0,\zeta)>0,\end{aligned} \tag{32}$$

for $\xi\geqslant h(0,0,\zeta)$. Further since $c_2(\xi,0,\zeta)>0$ for $\xi>h(0,0,\zeta)$ it follows from Sec. 6 of I that

$$X(\xi,\zeta)=8EFGH\,v(\xi)\int_{f_+(\xi,\zeta)}^{\infty}\frac{d\eta}{(\eta-x_2)[\overline{F}(\xi,\eta,\zeta)]^{1/2}}. \tag{33}$$

Note that

$$\overline{F}(\xi,x_2,0)=64E^2F^2G^2H^2[v(\xi)]^2F(\xi,x_2), \tag{34}$$

where $F(\xi,x_2)$ is given in Eq. (A1) (and in Eq. (I-12) with $\eta\to x_2$, $d\to X_4$).

Now since, as shown in Sec. 4, $\zeta_+(\xi,0,0)$ is the inverse of the strictly increasing function $h(0,0,\zeta)$ on $0\leqslant\zeta<\infty$ we find on inserting Eq. (33) into Eq. (25) and reversing the order of the ζ and ξ integrations that

$$\begin{aligned}I(x_i)&=\lim_{\epsilon\downarrow 0}\frac{\partial}{\partial K^2}\int_{h(0,0,\epsilon)}^{\infty}\frac{d\xi}{\xi-x_1}\int_{\epsilon}^{\zeta_+(\xi,0,0)}\frac{d\zeta}{\zeta}\\&\quad\times\int_{f_+(\xi,\zeta)}^{\infty}\frac{d\eta}{\eta-x_2}\frac{8EFGH\,v(\xi)}{[\overline{F}(\xi,\eta,\zeta)]^{1/2}}.\end{aligned} \tag{35}$$

Note that $h(0,0,\epsilon)$, $\zeta_+(\xi,0,0)$, and $\overline{F}(\xi,\eta,\zeta)$ depend on K^2 through Eqs. (4) and (9).

6. STUDY OF $f_+(\xi,\zeta)$

To reverse the order of the ζ and η integrations in Eq. (35), we need to examine the function $f_+(\xi,\zeta)$ for $\xi\geqslant 1$, $0\leqslant\zeta\leqslant\zeta_+(\xi,0,0)$. First we examine the behavior of $f_+(\xi,\zeta)$ as $\zeta\uparrow\zeta_+(\xi,0,0)$ with ξ fixed and $\geqslant 1$. We showed in Sec. 3 that when Eq. (10) holds, $l_1(0,\zeta)>0$ and thus from Eq. (22)

$$b_1(\xi,0,\zeta_+(\xi,0,0))<0. \tag{36}$$

Similarly

$$b_2(\xi,0,\zeta_+(\xi,0,0))<0 \tag{37}$$

and hence from Eqs. (18) and (31) and the fact that

$$v(\xi)>0 \tag{38}$$

for $\xi\geqslant 1$ it follows that $f_+(\xi,\zeta)\to+\infty$ as $\zeta\uparrow\zeta_+(\xi,0,0)$.

Next, from Eq. (31) we see that the derivative of $f_+(\xi,\zeta)$ with respect to ζ is

$$\begin{aligned}f_{+\zeta}(\xi,\zeta)&=[4GH\,v(\xi)]^{-1}[c_2(\xi,0,\zeta)]^{-2}\\&\quad\times(-b_1(\xi,0,\zeta)\{[b_2(\xi,0,\zeta)]^2-a_2(\xi)c_2(\xi,0,\zeta)\}^{1/2}\\&\quad-b_2(\xi,0,\zeta)\{[b_1(\xi,0,\zeta)]^2-a_1(\xi)c_1(\xi,0,\zeta)\}^{1/2})\\&\quad\times L(\xi,\zeta),\end{aligned} \tag{39}$$

where

$$L(\xi,\zeta)=[Q(\xi,\zeta)/\sqrt{R(\xi,\zeta)}]+[Q'(\xi,\zeta)/\sqrt{R'(\xi,\zeta)}], \tag{40}$$

$$\begin{aligned}Q(\xi,\zeta)&=\zeta(a_3b_1-\gamma b_3)+(b_3b_1-\gamma c_3),\\Q'(\xi,\zeta)&=\zeta(a_3b_2-\delta b_3)+(b_3b_2-\delta c_3),\\R(\xi,\zeta)&=\zeta^2(\gamma^2-a_3a_1)+2\zeta(\gamma b_1-a_1b_3)+b_1^2-a_1c_1,\\R'(\xi,\zeta)&=\zeta^2(\delta^2-a_3a_2)+2\zeta(\delta b_2-a_2b_3)+b_2^2-a_2c_2.\end{aligned} \tag{41}$$

In Eq. (41) a_1 has been written for $a_1(\xi)$, c_1 for $c_1(\xi,0,0)$, γ for $\gamma(\xi,-X_3)$ etc. where these quantities are defined in Eqs. (19) and (20). From Eq. (36) together with the facts that $b_1\equiv b_1(\xi,0,0)<0$ when Eq. (10) holds and $b_1(\xi,0,\zeta)$ is linear in ζ it follows that

$$b_1(\xi,0,\zeta)<0 \tag{42}$$

for all $0\leqslant\zeta\leqslant\zeta_+(\xi,0,0)$. Similarly

$$b_2(\xi,0,\zeta)<0, \tag{43}$$

for all $0 \leqslant \zeta \leqslant \zeta_+(\xi, 0, 0)$. Thus the term in boldface parentheses in Eq. (39) is always positive and $f_{+\zeta}(\xi, \zeta)$ vanishes if and only if $L(\xi, \zeta)$ vanishes.

Now

$$(b_3^2 - a_3c_3)R(\xi, \zeta) = Q^2(\xi, \zeta) - c_2(\xi, 0, \zeta)C_2(\xi, -X_3),$$
$$(b_3^2 - a_3c_3)R'(\xi, \zeta) = Q'^2(\xi, \zeta) - c_2(\xi, 0, \zeta)C_3(\xi, -e), \tag{44}$$

where $(b_3^2 - a_3c_3) > 0$ for $\xi \geqslant 1$,

$$C_2(\xi, -X_3) = c_3(\gamma^2 - a_1a_3) + b_1^2a_3 + b_3^2a_1 - 2\gamma b_1b_3,$$
$$C_3(\xi, -e) = c_3(\delta^2 - a_2a_3) + b_2^2a_3 + b_3^2a_2 - 2\delta b_2b_3, \tag{45}$$

and $c_2(\xi, 0, \zeta) > 0$ for $\xi > 1$, $0 \leqslant \zeta < \zeta_+(\xi, 0, 0)$. In Eqs. (44) and (45) the abbreviations described after Eq. (41) have again been used. The argument of Sec. 4 of VF then shows that $f_{+\zeta}(\xi, \zeta_0(\xi)) = 0$ with $0 \leqslant \zeta_0(\xi) < \zeta_+(\xi, 0, 0)$ if and only if

$$\text{(i) } C_2(\xi, -X_3) < 0, \quad C_3(\xi, -e) < 0 \tag{46}$$

and

$$\text{(ii) } Q(\xi, \zeta_0(\xi))/\sqrt{-C_2(\xi, -X_3)} = -Q'(\xi, \zeta_0(\xi))/\sqrt{C_3(\xi, -e)} \tag{47}$$

solves to give $0 \leqslant \zeta_0(\xi) < \zeta_+(\xi, 0, 0)$. Thus, for fixed $\xi \geqslant 1$ $f_+(\xi, \zeta)$ is strictly increasing on $0 \leqslant \zeta \leqslant \zeta_+(\xi, 0, 0)$ if and only if $L(\xi, 0) > 0$ or

$$\frac{G_2(\xi, -X_3)}{[\xi^2 + 2ab\xi + a^2 + b^2 - 1]^{1/2}} + \frac{G_3(\xi, -e)}{[\xi^2 + 2cX_4\xi + c^2 + X_4^2 - 1]^{1/2}} > 0, \tag{48}$$

where $G_2(\xi, -X_3)$ and $G_3(\xi, -e)$ are given by Eqs. (A13) and (A14). When Eq. (10) holds we see that each of the two terms in Eq. (48) is positive and hence $f_+(\xi, \zeta)$ increases strictly on $0 \leqslant \zeta \leqslant \zeta_+(\xi, 0, 0)$ for fixed $\xi \geqslant 1$.

7. SOLUTIONS OF $\bar{F}(\xi,\eta,\zeta) = 0$

Next we study the behavior of the zeros of $\bar{F}(\xi, \eta, \zeta)$ when ξ and η are held fixed. From Eqs. (30) and (19) we find that

$$\bar{F}(\xi, \eta, \zeta) = A(\xi, \eta; K^2)\zeta^2 + 2B(\xi, \eta; K^2)\zeta + C(\xi, \eta), \tag{49}$$

where

$$A(\xi, \eta; K^2) = a_3([\beta(\xi, \eta)]^2 - a_1a_2) + \gamma^2a_2 + \delta^2a_1 - 2\beta(\xi, \eta)\gamma\delta,$$
$$B(\xi, \eta; K^2) = b_3([\beta(\xi, \eta)]^2 - a_1a_2) + b_1\gamma a_2 + b_2\delta a_1 - \beta(\xi, \eta)b_1\delta - \beta(\xi, \eta)b_2\gamma,$$
$$C(\xi, \eta) = c_3([\beta(\xi, \eta)]^2 - a_1a_2) + b_1^2a_2 + b_2^2a_1 - 2\beta(\xi, \eta)b_1b_2. \tag{50}$$

The abbreviations described after Eq. (41) have again been used except for $\beta(\xi, \eta)$, which is the only term that depends on η.

The discriminant of the quadratic function of ζ in Eq. (49) is

$$\begin{aligned}[B(\xi, \eta; K^2)]^2 - A(\xi, \eta; K^2)C(\xi, \eta) = (&\{\beta(\xi, \eta)\}^2 - a_1a_2) \\ \times((&b_3^2 - a_3c_3)\{[\beta(\xi, \eta)]^2 - a_1a_2\} \\ &+ 2[-b_3(b_1\delta + b_2\gamma) + a_3b_1b_2 + c_3\gamma\delta]\beta(\xi, \eta) \\ &+ (b_1\delta - b_2\gamma)^2 + 2b_3(b_1\gamma a_2 + b_2\delta a_1) \\ &- c_3(\gamma^2a_2 + \delta^2a_1) - a_3(b_1^2a_2 + b_2^2a_1))\end{aligned} \tag{51}$$

and the term in boldface parentheses vanishes when

$$\beta(\xi.\eta) = (b_3^2 - a_3c_3)^{-1}\{b_3(b_1\delta + b_2\gamma) - a_3b_1b_2 - c_3\gamma\delta \pm [C_2(\xi, -X_3)C_3(\xi, -e)]^{1/2}\} \tag{52}$$

giving $\eta = p_\pm(\xi; K^2)$, with $p_\pm(\xi; K^2)$ defined in Eq. (A6). The terms $C_2(\xi, -X_3)$ and $C_3(\xi, -e)$ are defined in Eq. (45). Thus

$$[B(\xi, \eta; K^2)]^2 - A(\xi, \eta; K^2)C(\xi, \eta) = \{[\beta(\xi, \eta)]^2 - a_1a_2\}(16)^2E^2F^2G^2H^2K^2[v(\xi)]^3\, P(\xi, \eta; K^2), \tag{53}$$

where

$$P(\xi, \eta; K^2) = (\xi^2 + 2dX_5\xi + d^2 + X_5^2 - 1)(\eta - p_+(\xi; K^2)) \times(\eta - p_-(\xi; K^2)). \tag{54}$$

The discriminant in Eq. (53) is always nonnegative sinc the inverse of $f_+(\xi, \zeta)$ is real. To show that it is in fact positive we note first that when Eq. (10) holds

$$\{[\beta(\xi, \eta)]^2 - a_1(\xi)a_2(\xi)\} > 0 \tag{55}$$

for all $\xi \geqslant 1$, $\eta \geqslant f_+(\xi, 0)$ [or equivalently for $\eta \geqslant 1$, $\xi \geqslant g_+(\eta)$ where $g_+(\eta)$, defined in Eq. (A3) and in Eq. (I-36) with $d \to X_4$, is the inverse of $f_+(\xi, 0)$]. Establishing Eq. (55) is straightforward but tedious. Secondly in Appendix A we show that either $p_\pm(\xi; K^2)$ are complex conjugates or

$$p_-(\xi; K^2) \leqslant p_+(\xi; K^2) < f_+(\xi, 0), \tag{56}$$

for $\xi \geqslant 1$. Thus, since the first factor in Eq. (54) is positive when Eq. (10) holds and Eq. (38) is satisfied it follows that $P(\xi, \eta; K^2)$ and hence the right-hand side of Eq. (53) is in fact positive for $\xi \geqslant 1$, $\eta \geqslant f_+(\xi, 0)$.

The two real solutions of

$$\bar{F}(\xi, \eta, \zeta) = 0 \tag{57}$$

are

$$\zeta_{\substack{a\\b}}(\xi, \eta; K^2) = [A(\xi, \eta; K^2)]^{-1}(-B(\xi, \eta; K^2) \mp \{[B(\xi, \eta; K^2)]^2 - A(\xi, \eta; K^2)C(\xi, \eta)\}^{1/2}). \tag{58}$$

Now from Eqs. (30) and (49) it follows that

$$C(\xi, f_+(\xi, 0)) = 0 \tag{59}$$

and from Eqs. (50), (40), (41), (42), and (43)

$$B(\xi, f_+(\xi, 0); K^2) = [c_3]^{-2}\{b_1(R'(\xi, 0))^{-1/2} + b_2(R(\xi, 0))^{-1/2}\}L(\xi, 0) < 0. \tag{60}$$

Thus

$$\zeta_a(\xi, f_+(\xi, 0); K^2) = 0 \neq \zeta_b(\xi, f_+(\xi, 0); K^2)$$

and also as $\eta \to +\infty$

$$\zeta_{\substack{a\\b}}(\xi, f_+(\xi, 0); K^2) \sim \zeta_\pm(\xi, 0, 0),$$

where $\zeta_\pm(\xi, 0, 0)$ are defined by the right-hand side of Eq. (21) with $1 \to 3$, $\mu \to 0$, $\zeta \to 0$. It now follows that $\zeta_a(\xi, \eta; K^2)$ is the inverse of the strictly increasing function $f_+(\xi, \zeta)$ on $0 \leqslant \zeta \leqslant \zeta_+(\xi, 0, 0)$. Hence $\zeta_a(\xi, \eta, K^2)$ increases from 0 to $\zeta_+(\xi, 0, 0)$ as η increases from $f_+(\xi, 0)$ to $+\infty$.

8. SPECTRAL REPRESENTATION OF THE PENTAGON DIAGRAM AMPLITUDE

Since for fixed $\xi \geq 1$ $f_+(\xi, \zeta)$ is strictly increasing on $0 \leq \zeta \leq \zeta_+(\xi, 0, 0)$ and $\zeta_a(\xi, \eta; K^2)$ is the inverse of $f_+(\xi, \zeta)$ on $0 \leq \zeta \leq \zeta_+(\xi, 0, 0)$, Eq. (35) can be written

$$I(x_i) = \lim_{\epsilon \downarrow 0} \frac{\partial}{\partial K^2} \int_{h(0,0,\epsilon;K^2)}^{\infty} \frac{8EFGH\, v(\xi)\, d\xi}{\xi - x_1} \int_{f_+(\xi,\epsilon;K^2)}^{\infty} \frac{d\eta}{\eta - x_2} \times \Gamma(\xi, \eta, \epsilon; K^2), \tag{61}$$

where

$$\Gamma(\xi, \eta, \epsilon; K^2) = \int_{\epsilon}^{\zeta_a(\xi,\eta;K^2)} \frac{d\zeta}{\zeta[A(\xi,\eta;K^2)\zeta^2 + 2B(\xi,\eta;K^2)\zeta + C(\xi,\eta)]^{1/2}}$$

and $A(\xi, \eta; K^2)$, $B(\xi, \eta; K^2)$, and $C(\xi, \eta)$ are given in Eq. (50). Note that in Eq. (61) $h(0, 0, \epsilon)$ and $f_+(\xi, \epsilon)$ depend on K^2. From Eqs. (30) and (49) and the fact that $f_+(\xi, \zeta)$ is strictly increasing on $0 \leq \zeta \leq \zeta_+(\xi, 0, 0)$ we have $C(\xi, \eta) > 0$ for $\xi > 1$, $\eta > f_+(\xi, 0)$. The integration in Eq. (61) can then be performed (c.f. Sec. 5 of VF) to give

$$\Gamma(\xi, \eta, \epsilon; K^2) = [C(\xi, \eta)]^{-1/2} \times \ln \frac{C(\xi,\eta) + \epsilon B(\xi,\eta;K^2) + [C(\xi,\eta)]^{1/2}[\bar{F}(\xi,\eta,\epsilon;K^2)]^{1/2}}{\epsilon\{[B(\xi,\eta;K^2)]^2 - A(\xi,\eta;K^2)C(\xi,\eta)\}^{1/2}}. \tag{63}$$

The method of differentiating with respect to K^2 and taking the limit $\epsilon \downarrow 0$ is now very similar to that given in Sec. 6 (and 7) of I and in Ref. 11. We find that

$$I(x_i) = \int_1^{\infty} \frac{d\xi}{\xi - x_1} \int_{f_+(\xi,0)}^{\infty} \frac{d\eta}{\eta - x_2} \frac{8EFGH\, v(\xi)}{[C(\xi,\eta)]^{1/2}} \times \frac{(-\frac{1}{2})(\partial/\partial K^2)\{[B(\xi,\eta;K^2)] - A(\xi,\eta;K^2)C(\xi,\eta)\}}{\{[B(\xi,\eta;K^2)]^2 - A(\xi,\eta;K^2)C(\xi,\eta)\}}. \tag{64}$$

From Eqs. (8) and (19) it follows that the factor $\{[\beta(\xi,\eta)]^2 - a_1 a_2\}[v(\xi)]^3$ in Eq. (53) does not depend on K^2. Thus

$$I(x_i) = -\frac{1}{2}\int_1^{\infty} \frac{d\xi}{\xi - x_1} \int_{f_+(\xi)}^{\infty} \frac{d\eta}{\eta - x_2} \frac{1}{\sqrt{F(\xi,\eta)}} \times \frac{(\partial/\partial K^2)[K^2 P(\xi,\eta;K^2)]}{K^2 P(\xi,\eta;K^2)}$$

$$= -\frac{1}{2}\int_1^{\infty} \frac{d\xi}{\xi - x_1} \int_{f_+(\xi)}^{\infty} \frac{d\eta}{\eta - x_2} \frac{1}{\sqrt{F(\xi,\eta)}} \times \frac{(\partial/\partial K^2)\tilde{P}(\xi,\eta;K^2)}{\tilde{P}(\xi,\eta;K^2)}, \tag{65}$$

where

$$\tilde{P}(\xi, \eta; K^2) = 32E^2F^2G^2H^2K^2 P(\xi, \eta; K^2) \tag{66}$$

and $P(\xi, \eta; K^2)$ is given in Eq. (54). The functions $f_+(\xi)$ $(\equiv f_+(\xi, 0))$ and $F(\xi, \eta)$ are defined in Eqs. (A2) and (A1) [and in Eqs. (I-11) and (I-12) with $d \to X_4$]. Their properties are studied in detail in Sec. 8 of I. Note that the relationship between $\tilde{P}(\xi, \eta; K^2)$ and $P(\xi, \eta; K^2)$ is similar to that between $\tilde{F}(v, w)$ and $F(\xi, \eta)$ in Eq. (I-12); that is, $\tilde{P}(\xi, \eta; K^2)$ would be the function we would choose to describe the leading Landau curve of the pentagon diagram amplitude had we been working directly in the masses and kinematic invariants rather than in the related quantities in Eqs. (4) and (9).

As discussed in Sec. 2, while Eq. (10) can be satisfied with physical invariants for sufficiently large internal masses, the spectral representation in Eq. (65) does not in general correspond to the physical amplitude since for the physical amplitude associated with the pentagon diagram in Fig. 1 X_1, X_4, and X_5 would in general be negative. To obtain the physical amplitude one might then start with Eq. (65) and do an analytic continuation in X_1, X_4, and X_5. Continuation in $x_1 (= -X_1)$ (and also in x_2) is straightforward since x_1 occurs only in the Cauchy kernel. The continuation in X_4 and X_5 is much more difficult since $F(\xi, \eta)$ depends on X_4 and $P(\xi, \eta; K^2)$ depends on both X_4 and X_5. The inner integration in Eq. (65) can, of course, be carried out, for example by using real and, if $p_\pm(\xi; K^2)$ are complex, complex partial fractions, to give a single integral representation of $I(x_i)$. Thus in principle it should be possible to generalize the method of analytic continuation used in II to apply to the pentagon diagram amplitude. In this way it should be possible to determine directly how and when complex triangle, box and pentagon singularities occur, resulting in a breakdown of even a single dispersion integral over a real domain. For the general mass case that we have been considering in this paper this would be a very difficult problem because of the increased number of singularities and their more complicated behavior. However, it is likely that this program can be carried out for some specific processes of physical interest. The method used in II could in principle also be generalized to obtain $I(x_i)$ for the case when the stability conditions a, b, c, d, $e > -1$, rather than just a, b, c, d, $e > 0$, hold.

Finally we compare our spectral representation in Eq. (65) with that given in Eq. (23) of Ref. 6. First note that $I(x_i)$ given in Eq. (65) is real and well defined since $f_+(\xi)$ is real when Eq. (10) holds and, from Eqs. (A1) and (A10), $F(\xi, \eta) > 0$ for $\xi > 1$, $\eta > f_+(\xi)$ and $P(\xi, \eta; K^2) > 0$ for $\xi \geq 1$, $\eta \geq f_+(\xi)$. Further for fixed ξ both $F(\xi, \eta)$ and $P(\xi, \eta; K^2)$ are quadratic functions of η and for fixed η they are quadratic functions of ξ. In comparison, in Eq. (23) of Ref. 6 it is assumed that $p_\pm(\xi; K^2)$ are always real whereas we show in Appendix A that they can in fact be complex for the case considered there. More important, the spectral representation in Eq. (23) of Ref. 6 is divergent, that is, infinity is obtained when the integration is carried out.

APPENDIX A

We collect here a number of results involving the various functions needed in the main body of the paper. It is assumed throughout that Eq. (10) holds. From Eqs. (34) and (30),

$$\begin{aligned} F(\xi, x_2) &\equiv F(\xi, x_2; a, b, c, X_4), \\ &= (\xi^2 - 1)(x_2^2 - 1) - 2(\xi - 1)(x_2 - 1)(aX_4 + bc) \\ &\quad - 2(\xi - 1)(a + c)(b + X_4) - 2(x_2 - 1)(a + b)(c + X_4) \\ &\quad + (aX_4 - bc)^2 - (a + b + c + X_4)^2, \\ &= (\xi^2 - 1)[x_2 - f_+(\xi)][x_2 - f_-(\xi)], \\ &= (x_2^2 - 1)[\xi - g_+(x_2)][\xi - g_-(x_2)], \end{aligned} \tag{A1}$$

where from Eqs. (31) and (19)

$$\begin{aligned} f_\pm(\xi) &\equiv f_\pm(\xi,0) \\ &= (\xi^2-1)^{-1}[(\xi-1)(aX_4+bc)+(a+b)(c+X_4) \\ &\quad \pm(\xi^2+2ab\xi+a^2+b^2-1)^{1/2}(\xi^2+2cX_4\xi+c^2+X_4^2-1)^{1/2}] \end{aligned} \tag{A2}$$

and

$$\begin{aligned} g_\pm(x_2) &= (x_2^2-1)^{-1}[(x_2-1)(aX_4+bc)+(a+c)(b+X_4) \\ &\quad \pm(x_2^2+2acx_2+a^2+c^2-1)^{1/2}(x_2^2+2bX_4x_2+b^2+X_4^2-1)^{1/2}]. \end{aligned} \tag{A3}$$

Note also that

$$\begin{aligned} (\xi^2-1)F(\xi,x_2) &= [G(\xi,x_2)]^2-\{\xi^2+2ab\xi+a^2+b^2-1\} \\ &\quad \times\{\xi^2+2cX_4\xi+c^2+X_4^2-1\}, \end{aligned} \tag{A4}$$

where

$$\begin{aligned} G(\xi,x_2) &\equiv G(\xi,x_2;a,b,c,X_4) \\ &= -x_2(\xi^2-1)+(\xi-1)(aX_4+bc)+(a+b)(c+X_4). \end{aligned} \tag{A5}$$

The above functions (with $X_4\to d$, $x_2\to y$) were also defined in Eqs. (I-12), (I-11), (I-36), and (I-A5) and their properties were discussed in detail in Sec. 8 and Appendix A of I and in Sec. 4 of II.

From Eqs. (52) and (45) we find that

$$\begin{aligned} p_\pm(\xi;K^2) &= (\xi^2+2dX_5\xi+d^2+X_5^2-1)^{-1} \\ &\quad \times[-E(\xi,X_3,e)\pm\{F_2(\xi,-X_3)F_3(\xi,-e)\}^{1/2}], \end{aligned} \tag{A6}$$

where

$$\begin{aligned} &E(\xi,X_3,e) \\ &= (\xi^2-1)eX_3+(\xi-1)[e(ad+bX_5)+X_3(cd+X_4X_5)-aX_4-bc] \\ &\quad +e(a+b)(X_5+d)+X_3(c+X_4)(X_5+d)+d^2ac+X_5^2bX_4 \\ &\quad -X_5d(bc+aX_4), \end{aligned} \tag{A7}$$

$$F_2(\xi,-X_3)\equiv F(\xi,-X_3;a,b,X_5,d), \tag{A8}$$

$$F_3(\xi,-e)\equiv F(\xi,-e;c,X_4,X_5,d), \tag{A9}$$

with $F(\xi,x_2;a,b,c,X_4)$ given in Eq. (A1).

Since

$$(\xi^2+2dX_5\xi+d^2+X_5^2-1)>0$$

the inequality

$$P(\xi,\eta;K^2)>0 \tag{A10}$$

will hold for $\xi\geq 1$, $\eta\geq f_+(\xi)$ if either $p_\pm(\xi;K^2)$ are complex conjugates or if Eq. (56) holds. That it is possible, when Eq. (10) holds, for $p_\pm(\xi;K^2)$ to be either real or complex conjugates depending on the value of ξ, where $\xi\geq 1$, can be seen as follows. Consider first the case when

$$0<a,b,X_5,d<1,\ X_3>0. \tag{A11}$$

Then one of the four possible configurations of the curve Γ defined by $F_2(\xi,-X_3)=0$ is as shown in Fig. 1 of II with $\eta\to -X_3$. We see that $F_2(\xi,-X_3)$ may be positive, zero or negative depending on the values of ξ and X_3. When a,b,X_5,d are no longer restricted to be less than 1, then there are more different configurations of Γ. Examples of the possible configurations of Γ are sketched in Ref. 12. Again $F_2(\xi,-X_3)$, and also $F_3(\xi,-e)$, may be positive, zero or negative. This statement is still true if the zeros on the right-hand sides of the inequalities in Eq. (10) are replaced by ones, the case initially considered in Ref. 6.

We now have the following cases to consider

(i) $F_2(\xi,-X_3)F_3(\xi,-e)<0$. Then $p_\pm(\xi;K^2)$ are complex conjugates and $P(\xi,\eta;K^2)>0$;

(ii) $F_2(\xi,-X_3)\geq 0$, $F_3(\xi,-e)\geq 0$. From Eq. (A7) it follows that

$$\begin{aligned} &(\xi^2-1)E(\xi,X_3,e)+(\xi^2+2dX_5\xi+d^2+X_5^2-1) \\ &\times[(\xi-1)(aX_4+bc)+(a+b)(c+X_4)] \\ &\quad = G_2(\xi,-X_3)G_3(\xi,-e), \end{aligned} \tag{A12}$$

where

$$G_2(\xi,-X_3)\equiv G(\xi,-X_3;a,b,X_5,d)>0, \tag{A13}$$

$$G_3(\xi,-e)\equiv G(\xi,-e;c,X_4,X_5,d)>0, \tag{A14}$$

and $G(\xi,x_2;a,b,c,X_4)$ is given in Eq. (A5). Then using two equations similar to Eq. (A4), relating $F_2(\xi,-X_3)$ and $G_2(\xi,-X_3)$ and relating $F_3(\xi,-e)$ and $G_3(\xi,-e)$, and defining

$$\cosh\kappa_1 = \frac{G_2(\xi,-X_3)}{(\xi^2+2dX_5\xi+d^2+X_5^2-1)^{1/2}(\xi^2+2ab\xi+a^2+b^2-1)^{1/2}}, \tag{A15}$$

$$\cosh\kappa_2 = \frac{G_3(\xi,-e)}{(\xi^2+2dX_5\xi+d^2+X_5^2-1)^{1/2}(\xi^2+2cX_4\xi+c^2+X_4^2-1)^{1/2}} \tag{A16}$$

we find that

$$\begin{aligned} p_\pm(\xi;K^2)-f_+(\xi) &= (\xi^2-1)^{-1}(\xi^2+2ab\xi+a^2+b^2-1)^{1/2} \\ &\quad \times(\xi^2+2cX_4\xi+c^2+X_4^2-1)^{1/2} \\ &\quad \times[-1-\cosh\kappa_1\cosh\kappa_2\pm\sinh\kappa_1\sinh\kappa_2] \\ &= (\xi^2-1)^{-1}(\xi^2+2ab\xi+a^2+b^2-1)^{1/2} \\ &\quad \times(\xi^2+2cX_4\xi+c^2+X_4^2-1)^{1/2} \\ &\quad \times[-1-\cosh(\kappa_1\mp\kappa_2)]<0. \end{aligned} \tag{A17}$$

Thus Eqs. (56) and (A10) hold.

(iii) $F_2(\xi,-X_3)<0$, $F_3(\xi,-e)<0$. In this case we define $\cos\phi_1$ by the right-hand side of Eq. (A15) and $\cos\phi_2$ by the right-hand side of Eq. (A16). Then

$$\begin{aligned} &p_\pm(\xi;K^2)-f_+(\xi) \\ &= (\xi^2-1)^{-1}(\xi^2+2ab\xi+a^2+b^2-1)^{1/2}(\xi^2+2cX_4\xi+c^2 \\ &\quad +X_4^2-1)^{1/2}[-1-\cos(\phi_1\pm\phi_2)]<0 \end{aligned} \tag{A18}$$

since the inequalities in Eqs. (A13) and (A14) hold. Again Eqs. (56) and (A10) are valid.

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[1]R. E. Cutkosky, Phys. Rev. Lett. 4, 624 (1960); J. Math. Phys. 1, 294 (1960).

[2]L.D. Landau, Nucl. Phys. 13, 181 (1959); Zh. Eksp. Teor. Fiz. 37, 62 (1959) [Sov. Phys. JETP 10, 45 (1960)].

[3]R.J. Eden, P.V. Landshoff, D.I. Olive, and J.C. Polkinghorne, *The Analytic S-Matrix* (Cambridge U.P., Cambridge, 1966), p. 115.

[4]L. F. Cook and J. Tarski, Phys. Rev. Lett. 5, 585 (1960); J. Math. Phys. 3, 1 (1962).

[5]F.R. Halpern, Phys. Rev. Lett. 10, 310 (1963).

[6]O.I. Zav'yalov and V.P. Pavlov, Zh. Eksp. Teor. Fiz. 44, 1500 (1963) [Sov. Phys. JETP 17, 1009 (1963)].

[7]J.S. Frederiksen and W.S. Woolcock, Nucl. Phys. B 28, 605 (1971).

[8]J.S. Frederiksen and W.S. Woolcock, Ann. Phys. (N.Y.) 75, 503 (1973).

[9]J.S. Frederiksen and W.S. Woolcock, Ann. Phys. (N.Y.) 80, 86 (1973).

[10]S.S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper and Row, New York, 1961).

[11]J.S. Frederiksen, Ph. D. thesis, Australian National University (unpublished); Bull. Austral. Math. Soc. 7, 455 (A) (1972).

[12]M. Fowler, P.V. Landshoff, and R.W. Lardner, Nuovo Cimento 17, 956 (1960).